

A smallness regularity criterion for the 3D Navier-Stokes equations in the largest class

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Abstract

In this paper, we consider the three-dimensional Navier-Stokes equations, and show that if the $\dot{B}_{\infty,\infty}^{-1}$ -norm of the velocity field is sufficiently small, then the solution is in fact classical.

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1. Introduction

Consider the following three-dimensional (3D) Navier-Stokes equations:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} - \Delta \mathbf{u} + \nabla \pi = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0, \end{cases} \quad (1)$$

where $\mathbf{u} = (u_1(x, t), u_2(x, t), u_3(x, t))$ is the fluid velocity, $\pi = \pi(x, t)$ is a scalar pressure; and \mathbf{u}_0 is the prescribed initial velocity field satisfying the compatibility condition $\nabla \cdot \mathbf{u}_0 = 0$.

The existence of a global weak solution

$$\mathbf{u} \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$$

to (1) has long been established by Leray [10], see also Hopf [9]. But the issue of regularity and uniqueness of \mathbf{u} remains open. Initialed by Serrin

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[15, 16] and Prodi [14], there have been a lot of literatures devoted to finding sufficient conditions to ensure \mathbf{u} to be smooth, see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 12, 13, 17, 18, 19, 20, 21, 22, 23] and references cited therein. Noticeably, the following Ladyzhenskaya-Prodi-Serrin condition ([6, 14, 15, 16]):

$$\mathbf{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \text{ with } \frac{2}{p} + \frac{3}{q} = 1, \quad 3 \leq q \leq \infty \quad (2)$$

can ensure the smoothness of the solution.

Note that the limiting case $L^\infty(0, T; L^3(\mathbb{R}^3))$ in (2) does not fall into the framework of standard energy method, which was proved by Escauriaza, Seregin and Šverák [6] using backward uniqueness theorem. Due to the fact that

$$L^3(\mathbb{R}^3) \subset \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3), \text{ but } L^3(\mathbb{R}^3) \neq \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3),$$

we shall consider in this paper the regularity of solutions of (1) in $\dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)$. However, we could not prove a regularity criterion as $L^\infty(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3))$, since the function in $\dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)$ has no decay at infinity, which ensures that the solution is smooth outside an big ball centered at origin so that the backward uniqueness theorem can be applied.

Before we state the precise result, let us recall the weak formulation of (1).

Definition 1. Let $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ satisfying $\nabla \cdot \mathbf{u}_0 = 0$, $T > 0$. A measurable vector-valued function \mathbf{u} defined in $[0, T] \times \mathbb{R}^3$ is said to be a weak solution to (1) if

1. $\mathbf{u} \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3));$
2. \mathbf{u} satisfies (1)_{1,2} in the sense of distributions;
3. \mathbf{u} satisfies the energy inequality:

$$\|\mathbf{u}(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla \mathbf{u}(s)\|_{L^2}^2 \, ds \leq \|\mathbf{u}_0\|_{L^2}^2, \quad \text{a.e. } t \in [0, T].$$

Now, our main result reads:

Theorem 2. *Let $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ satisfying $\nabla \cdot \mathbf{u}_0 = 0$, $T > 0$. Assume that \mathbf{u} is a weak solution of (1) in $[0, T]$. If there exists an absolute constant $\varepsilon_0 > 0$ such that*

$$\|\mathbf{u}\|_{\dot{B}_{\infty, \infty}^{-1}} \leq \varepsilon_0, \quad (3)$$

then \mathbf{u} is smooth in $(0, T)$.

The rest of this paper is organized as follows. In section 2, we recall the definition of Besov spaces and an interpolation inequality. Section 3 is devoted to proving Theorem 2.

2. Preliminaries

We first introduce the Littlewood-Paley decomposition. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. For $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-ix \cdot \xi} dx.$$

Let us choose an non-negative radial function $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that

$$0 \leq \hat{\varphi}(\xi) \leq 1, \quad \hat{\varphi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2, \end{cases}$$

and let

$$\psi(x) = \varphi(x) - 2^{-3}\varphi(x/2), \quad \varphi_j(x) = 2^{3j}\varphi(2^j x), \quad \psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}.$$

For $j \in \mathbb{Z}$, the Littlewood-Paley projection operators S_j and Δ_j are, respectively, defined by

$$S_j f = \varphi_j * f, \quad \Delta_j f = \psi_j * f.$$

Observe that $\Delta_j = S_j - S_{j-1}$. Also, it is easy to check that if $f \in L^2(\mathbb{R}^3)$, then

$$S_j f \rightarrow 0, \text{ as } j \rightarrow -\infty; \quad S_j f \rightarrow f, \text{ as } j \rightarrow \infty,$$

in the L^2 sense. By telescoping the series, we have the following Littlewood-Paley decomposition

$$f = \sum_{j=-\infty}^{\infty} \Delta_j f,$$

for all $f \in L^2(\mathbb{R}^3)$, where the summation is in the L^2 sense.

Let $s \in \mathbb{R}$; $p, q \in [1, \infty]$, the homogeneous Besov space $\dot{B}_{p,q}^s(\mathbb{R}^3)$ is defined by the full dyadic decomposition such as

$$\dot{B}_{p,q}^s = \left\{ f \in \mathcal{Z}'(\mathbb{R}^3); \|f\|_{\dot{B}_{p,q}^s} = \left\| \{2^{js} \|\Delta_j f\|_{L^p}\}_{j=-\infty}^{\infty} \right\|_{\ell^q} < \infty \right\},$$

where $\mathcal{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^3) = \left\{ f \in \mathcal{S}(\mathbb{R}^3); D^\alpha \hat{f}(0) = 0, \quad \forall \alpha \in \mathbb{N}^3 \right\}.$$

The following interpolatin inequality will be need in Section 3,

$$\|f\|_{L^q} \leq C \|f\|_{\dot{H}^{\alpha(\frac{q}{2}-1)}}^{\frac{2}{q}} \|f\|_{\dot{B}_{\infty,\infty}^{-\alpha}}^{1-\frac{2}{q}}, \quad \forall f \in \dot{H}^{\alpha(\frac{q}{2}-1)}(\mathbb{R}^3) \cap \dot{B}_{\infty,\infty}^{-\alpha}(\mathbb{R}^3), \quad (4)$$

where $2 < q < \infty$ and $\alpha > 0$. See [11] for the proof.

3. Proof of Theorem 2

In this section, we shall prove Theorem 2.

By the classical “weak=strong” type uniqueness theorem, we need only to derive the a priori estimate

$$\mathbf{u} \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)). \quad (5)$$

Multiplying (1)₁ by $-\Delta \mathbf{u}$, integrating over \mathbb{R}^3 , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 + \|\Delta \mathbf{u}\|_{L^2}^2 &= \int_{\mathbb{R}^3} [(\mathbf{u} \cdot \nabla) \mathbf{u}] \cdot \Delta \mathbf{u} \, dx \\ &\equiv I. \end{aligned} \quad (6)$$

By Hölder inequality,

$$I \leq \|\mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^3} \|\Delta \mathbf{u}\|_{L^2}.$$

Invoking (4) with $q = 6$, $\alpha = 1$; and $q = 3$, $\alpha = 2$, we may further estimate I as

$$\begin{aligned} I &\leq C \left(\|\mathbf{u}\|_{\dot{H}^2}^{\frac{1}{3}} \|\mathbf{u}\|_{\dot{B}_{\infty,\infty}^{-1}}^{\frac{2}{3}} \right) \left(\|\nabla \mathbf{u}\|_{\dot{H}^1}^{\frac{2}{3}} \|\nabla \mathbf{u}\|_{\dot{B}_{\infty,\infty}^{-2}}^{\frac{1}{3}} \right) \|\Delta \mathbf{u}\|_{L^2} \\ &= C \|\mathbf{u}\|_{\dot{B}_{\infty,\infty}^{-1}} \|\Delta \mathbf{u}\|_{L^2}^2. \end{aligned} \quad (7)$$

Substituting (7) into (6), we see

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 + \left(1 - C \|\mathbf{u}\|_{\dot{B}_{\infty,\infty}^{-1}} \right) \|\Delta \mathbf{u}\|_{L^2}^2 \leq 0.$$

Thus, if

$$\|\mathbf{u}\|_{\dot{B}_{\infty,\infty}^{-1}} \leq \frac{1}{C} \equiv \varepsilon_0,$$

we deduce that $\|\nabla \mathbf{u}\|_{L^2}$ is decreasing, and thus (5), as desired.

The proof of Theorem 2 is completed.

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